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REDUCED BODIES IN NORMED PLANES

BY

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ABSTRACT

We say that a convex body R of a d-dimensional real normed linear space M^d is **reduced**, if $\Delta(P) < \Delta(R)$ for every convex body $P \subset R$ different from R. The symbol $\Delta(C)$ stands here for the thickness (in the sense of the norm) of a convex body $C \subset M^d$. We establish a number of properties of reduced bodies in M^2. They are consequences of our basic Theorem which describes the situation when the width (in the sense of the norm) of a reduced body $R \subset M^2$ is larger than $\Delta(R)$ for all directions strictly between two fixed directions and equals $\Delta(R)$ for these two directions.

1. Introduction

A convex body R in d-dimensional Euclidean space E^d, is called **reduced** provided every convex body properly contained in R has thickness smaller than the thickness of R. Of course, every body of constant width in E^d is reduced. Other simple examples of reduced bodies are regular odd-gongs in E^2. The notion of reduced body was introduced in [7]. A number of properties of reduced bodies in Euclidean space are given in [3], [4], [6], [7], [9], [10], [11], and [14]. For the context see [8, Section 5] and [15, Section 2].

In [13] the notion of reduced body is extended to the d-dimensional real normed linear space M^d (called also d-dimensional Banach space or Minkowski space) and some general properties of reduced bodies in M^d are given. An origin-symmetric convex body D in R^n is called the **unit ball** of M^d and the function $\|x\| = \min\{\lambda \geq 0 : x \in \lambda D\}$ determines the **norm** of M^d. For basic results on d-dimensional Banach spaces from the point of view of functional analysis see the survey [5], for more geometric aspects we refer to the book [16].

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In the paper [12] reduced polytopes in M^d are investigated. The aim of the present paper is to give further properties of reduced bodies in M^2.

The terminology and notation of our paper is consistent with [9] and [13]. Nevertheless, for the convenience of the reader we recall necessary notions for $d = 2$. If H_1 and H_2 are different parallel lines in M^2, then the convex hull $S = \mathrm{conv}(H_1 \cup H_2)$ is called a **strip**. If these lines are perpendicular (in Euclidean sense) to a direction m, then S is called a **strip of direction** m. We call H_1 and H_2 the bounding lines of S. If both H_1 and H_2 are supporting lines of a convex body C, then we say that S is a C**-strip**. The C**-strip of direction** m is denoted by $S(C, m)$. If a point x of the boundary $\mathrm{bd}(C)$ of C belongs to one of the bounding lines of a C-strip S, we say that S **passes through** x. By ab we denote the segment with endpoints $a, b \in M^2$ and by $|ab|$ we denote the M^2**-distance** $\|b - a\|$ of a and b. We call a segment ab **degenerate** if $a = b$. Any degenerate segment is considered to be parallel to every segment. By the M^2**-width of a strip** $S = \mathrm{conv}(H_1 \cup H_2)$ we mean the diameter of the largest ball contained in S. Observe that this is nothing else than the smallest value of $|x_1x_2|$ over all $x_1 \in H_1$ and $x_2 \in H_2$. We consider the width of a convex body $C \subset M^2$ in the sense of the norm: by the M^2**-width** $w(C, m)$ of C in direction m we understand the M^2-width of $S(C, m)$. The minimum of $w(C, m)$ taken over all directions is called the M^2**-thickness** of C or the **thickness in the sense of the norm**, and it is denoted by $\Delta(C)$. If a chord ab of a convex body $C \subset M^2$ connects the opposite lines bounding a C-strip of M^2-thickness $\Delta(C)$ and if $|ab| = \Delta(C)$, it is called a **thickness chord of** C.

We say that a convex body $R \subset M^2$ is **reduced**, if $\Delta(P) < \Delta(R)$ for every convex body $P \subset R$ different from R.

For many extremal problems concerning the thickness and, more generally, the M^2-thickness of convex bodies it is sufficient to consider only reduced bodies. This makes the subject important. For such applications of reduced bodies in M^2, and also in M^d, see [1] and [2].

If the oriented positive angle from a direction m_1 to a different direction m_2 in M^2 is smaller than π, then we write $m_1 \prec m_2$. If $m_1 \prec m_2$ or $m_1 = m_2$, we write $m_1 \preceq m_2$. If $m_1 \preceq m_2$ and $m_1 \preceq m \preceq m_2$, we say that m is **between** m_1 and m_2. Similarly, we say that m is **strictly between** m_1 and m_2 provided $m_1 \prec m_2$ and $m_1 \prec m \prec m_2$. The set of directions m which are strictly between two fixed directions is called **an open interval of directions**.

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2. Basic result

Let $C \subset M^2$ be a convex body and let a C-strip $S(C, m)$ have M^2-width $\Delta(C)$. It is easy to see that *the union of all thickness chords of C which connect the straight lines bounding $S(C, m)$ is a trapezium $T(C, m)$ whose bases are in the mentioned straight lines.* Denote these bases of $T(C, m)$ by $A(C, m)$ and $B(C, m)$ such that $A(C, m)$ is in this straight line bounding $S(C, m)$ for which the outer normal of $S(C, m)$ has direction m. Of course it can happen that one or both of the segments $A(C, m)$, $B(C, m)$ are non-degenerate segments.

By **the first (last) point of a segment** lying in the boundary of a convex body we mean the first (last) point of this segment reached when we go counterclockwise on the boundary of the body.

Our basic result, the following Theorem, generalizes Theorem 3 from [9]. Similarly as in the Euclidean case considered in [9], our Theorem enables to derive a few interesting facts about reduced bodies in M^2. They are presented in the next section.

THEOREM: *Let R be a reduced body in M^2. Assume that m_1 and m_2 are directions satisfying $m_1 \prec m_2$ such that $w(R, m_1) = \Delta(R) = w(R, m_2)$ and that $w(R, m) > \Delta(R)$ for every direction m strictly between m_1 and m_2. Denote by a_1 the last point of $A(R, m_1)$, by a_2 the first point of $A(R, m_2)$, by b_1 the last point of $B(R, m_1)$ and by b_2 the first point of $B(R, m_2)$. We claim that the segments a_1a_2 and b_1b_2 are non-degenerate and that they are in* $\mathrm{bd}(R)$. *Moreover, one of these two segments is perpendicular to m_1 and the other is perpendicular to m_2.*

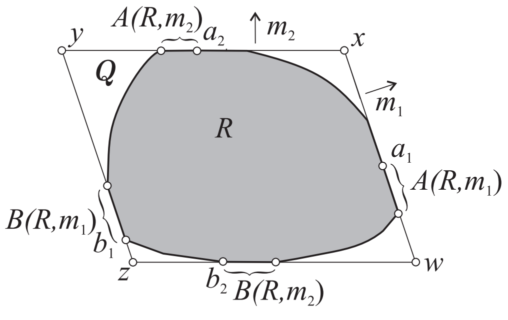

Figure 1

Proof: Consider the parallelogram $Q = S(R, m_1) \cap S(R, m_2)$ with consecutive vertices w, x, y, z in the positive orientation; the notation is chosen so that $A(R, m_1) \subset wx$, $A(R, m_2) \subset xy$, $B(R, m_1) \subset yz$ and $B(R, m_2) \subset zw$. The assumptions of the theorem are illustrated in Figure 1, where the parallelogram Q is also presented. The thesis of the theorem is illustrated in Figures 3 and 4.

Let $g_1 \in yz$ and $g_2 \in zw$ be the closest points to z such that a_1g_1 and a_2g_2 are thickness chords of R. Similarly, let $h_1 \in wx$ and $h_2 \in xy$ be the closest points to x such that b_1h_1 and b_2h_2 are thickness chords of R.

We divide the proof into Parts 1–6.

Part 1: We intend to show that if $a_1 \neq a_2$, then the oriented piece $\widehat{a_1a_2}$ of $\mathrm{bd}(R)$ from a_1 to a_2 is a line segment. Assume the opposite:

(1) $$\widehat{a_1a_2} \quad \text{is not a line segment.}$$

Denote by a_1' the closest point to x from the intersection of R with wx, and by a_2' the closest point to x from the intersection of R with xy. Due to assumption (1), there exists an extreme point $e \in \widehat{a_1a_2}$ of R different from a_1 and a_2. Moreover, by Theorem 1 from [13] there is an R-strip of width $\Delta(R)$ which passes through e. From these facts we conclude that if $e \notin a_1x$ and $e \notin xa_2$, then this R-strip has the form $S(R, m)$, where m is strictly between m_1 and m_2. This contradicts the assumption of the theorem that $w(R, m) > \Delta(R)$ for every m strictly between m_1 and m_2. Thus the only case to be considered is the case when $\widehat{a_1a_2}$ is the union of segments a_1a_1', $a_1'a_2'$, and $a_2'a_2$. Of course, at least one of the boundary segments a_1a_1', $a_2'a_2$ of R is non-degenerate.

Consider the mentioned case.

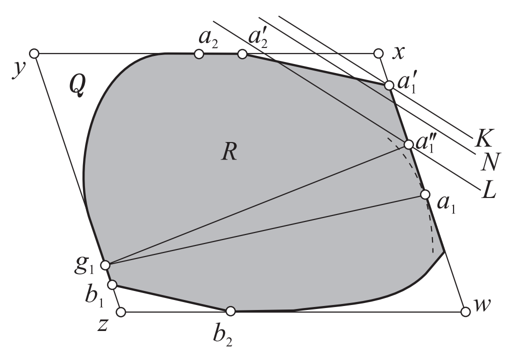

Figure 2

For instance, assume that $a_1 \neq a_1'$ (see Figure 2). Let K denote this straight line parallel to the segment a_1a_2 which passes through a_1'. Let a_1'' be the midpoint of the segment a_1a_1'. Since $g_1 \in yz$ and $a_1'' \notin A(R, m_1)$, we get

$|a_1''g_1| > \Delta(R)$ from the definition of a_1. Now we provide the straight line L parallel to K which passes through a_1''. Define R_L as the intersection of R with the closed halfplane bounded by L and containing a_1a_2. Take the disk $F = \{x \in M^2 : \|x - g_1\| \leq \Delta(R)\}$. From the definitions of a_1 and g_1 we see that $\mathrm{bd}(F)$ intersects the segment a_1a_1'' only at a_1 (a piece of $\mathrm{bd}(F)$ is marked by a broken line in Figure 2). This and $g_1 \in R$ imply that there is a direction l_1 strictly between m_1 and m_2 such that

$$w(R_L, m) \geq \Delta(R) \quad \text{for every direction } m \text{ satisfying } m_1 \preceq m \preceq l_1.$$

Analogously, if $a_2 \neq a_2'$, then there is a direction l_2 strictly between m_1 and m_2 such that

$$w(R_L, m) \geq \Delta(R) \quad \text{for every direction } m \text{ satisfying } l_2 \preceq m \preceq m_2.$$

The last statement holds true also when $a_2' = a_2$, where in the place of l_2 we take this direction perpendicular to the segment $a_1'a_2$ which is strictly between m_1 and m_2. In order to see this, first note that assumption (1) implies $a_1' \neq x$. Moreover, (1) and $a_2' = a_2$ imply that $a_1'a_2$ is a non-degenerate segment. Observe that L intersects this segment in an interior point, c. Thus the segment ca_2 is contained in $\mathrm{bd}(R)$ and in $\mathrm{bd}(R_L)$. This confirms the statement formulated at the beginning of this paragraph.

Clearly, we may assume that $l_1 \prec l_2$. So $m_1 \prec l_1 \prec l_2 \prec m_2$. Of course, $w(R, l_2) > \Delta(R)$.

Recapitulating the above considerations, we see that there are directions l_1 and l_2 such that $m_1 \prec l_1 \prec l_2 \prec m_2$ and that

$$(2) \quad w(R_L, m) \geq \Delta(R) \quad \text{for every } m \text{ satisfying } m_1 \preceq m \preceq l_1 \text{ or } l_2 \preceq m \preceq m_2.$$

There is an $\varepsilon > 0$ such that

$$(3) \qquad w(R, m) \geq \Delta(R) + \varepsilon \quad \text{for every } m \text{ satisfying } l_1 \preceq m \preceq l_2.$$

The reason is that otherwise a sequence $\{n_i\}$ of directions between l_1 and l_2 would exist such that $\lim_{i\to\infty} w(R, n_i) = \Delta(R)$, and thus, by compactness arguments, a direction n between l_1 and l_2 would exist such that $w(R, n) = \Delta(R)$. This would contradict the assumption of the theorem that $w(R, m) > \Delta(R)$ for every m strictly between m_1 and m_2.

Let N be a straight line parallel to K lying strictly between K and L such that the M^2-thickness ε_N of the strip bounded by K and N is at most ε (see Figure 2).

The line N dissects R into two closed subsets. That one containing R_L is denoted by R_N. Observe that

$$w(R_N, m) \geq w(R, m) - \varepsilon_N \quad \text{for every direction } m. \tag{4}$$

From (2), (3) and (4), from $\varepsilon_N \leq \varepsilon$ and from $R_L \subset R_N \subset R$ we conclude that $\Delta(R_N) = \Delta(R)$. This contradicts the fact that R is reduced. Thus our assumption (1) is false, which means that $\widehat{a_1 a_2}$ is nothing else than the segment $a_1 a_2$.

PART 2: Following the same steps as above with respect to the opposite vertex z of Q, we obtain that if $b_1 \neq b_2$, then $\widehat{b_1 b_2}$ is the segment $b_1 b_2$.

PART 3: We are planning to show that at least one of the points a_1, a_2 coincides with x.

For this purpose we assume in Part 3 the opposite situation: $a_1 \neq x$ and $a_2 \neq x$. Since $a_1 \in wx$, $a_2 \in xy$, $a_1 \neq x$ and $a_2 \neq x$, we have $a_1 \neq a_2$.

First we will show that $a_1 = h_1$. Suppose that $a_1 \neq h_1$. Observe that by the definition of h_1 we have $h_1 \in wa_1$. Let T denote the straight line parallel to the segment $h_1 a_2$ which passes through a_1. From Part 1 we see that $\widehat{h_1 a_2}$ is nothing else than the union of non-degenerate segments $h_1 a_1$ and $a_1 a_2$. Let k_2 denote this direction perpendicular to $a_1 a_2$ which satisfies $m_1 \preceq k_2 \preceq m_2$.

Let $\hat{a}_1$ be the midpoint of the segment $h_1 a_1$. Denote by U the straight line parallel to T passing through $\hat{a}_1$. The intersection of R with the halfplane bounded by U and containing $h_1 a_2$ is denoted by R_U. Since h_1 is the point of wx closest to x such that $b_1 h_1$ is a thickness chord of R and since $a_1 \neq h_1$, we have $|b_1 a_1| > \Delta(R)$ and $|b_1 \hat{a}_1| > \Delta(R)$. Thus there is a direction k_1 strictly between m_1 and k_2 such that $w(R_U, m) \geq \Delta(R)$ for every direction m between m_1 and k_1. Even more, by the definition of k_2 we have the following:

$$\text{if } m_1 \preceq m \preceq k_1 \text{ or } k_2 \preceq m \preceq m_2, \text{ then } w(R_U, m) \geq \Delta(R). \tag{5}$$

There is a $\delta > 0$ such that

$$w(R, m) \geq \Delta(R) + \delta \quad \text{for every direction } m \text{ satisfying } k_1 \preceq m \preceq k_2. \tag{6}$$

Otherwise, there would exists a sequence $\{n_i\}$ of directions between k_1 and k_2 such that $\lim_{i\to\infty} w(R, n_i) = \Delta(R)$, and thus, by compactness arguments, there would exist a direction n between k_1 and k_2 such that $w(R, n) = \Delta(R)$. This would contradict the assumption of the theorem.

Denote by V a straight line parallel to U which lies strictly between U and T such that the M^2-thickness δ_V of the strip bounded by V and U is at most δ (the reader is asked to draw a figure analogous to Figure 2). The closed part of R bounded by V and containing R_T is denoted by R_V. We have

$$w(R_V, m) \geq w(R, m) - \delta_V \quad \text{for every direction } m. \tag{7}$$

From (5),(6) and (7), from $\delta_V \leq \delta$ and from the inclusions $R_U \subset R_V \subset R$ we obtain $\Delta(R_V) = \Delta(R)$. Since R_V is a convex body contained in R, this equality contradicts the fact that R is reduced. Consequently, $a_1 = h_1$.

Following the same steps, we obtain that $a_2 = h_2$.

Since $a_1 = h_1$ and $a_2 = h_2$, we have $|a_1b_1| = \Delta(R)$ and $|a_2b_2| = \Delta(R)$, which means that a_1b_1 and a_2b_2 are thickness chords of R. Since $\widehat{a_1a_2}$ and $\widehat{b_1b_2}$ are segments, we see that at least one of the thickness chords a_1b_1, a_2b_2 of R has both its endpoints in the lines bounding $S(R, k_2)$. From these facts we conclude that $\Delta(S(R, k_2)) = \Delta(R)$. Since $m_1 \preceq k_2 \preceq m_2$, this contradicts the assumption of the theorem that $w(R, m) > \Delta(R)$ for every direction m strictly between m_1 and m_2. Thus at least one of the points a_1, a_2 coincides with x (see Figures 3 and 4).

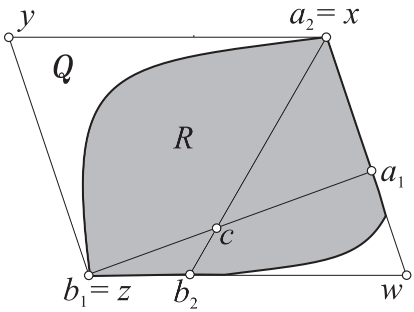

Figure 3

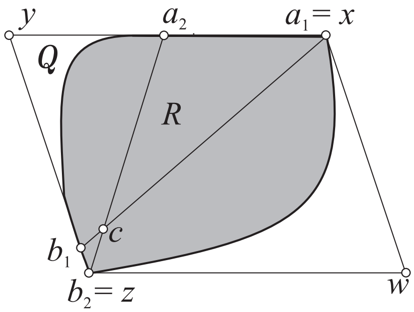

Figure 4

Part 4: Repeating the considerations of Part 3 with respect to the vertex z instead of x, we show that at least one of the points b_1, b_2 coincides with z (see Figures 3 and 4).

Part 5: Let us explain why the segments a_1a_2 and b_1b_2 are not parallel. Suppose the opposite, i.e., that a_1a_2 and b_1b_2 are parallel. From this assumption and since at least one of the points a_1, a_2 coincides with x and at least one of the points b_1, b_2 coincides with z (as shown in Part 3 and 4), we obtain that at least one of the segments a_1b_1, a_2b_2 coincides with the segment xz. Moreover, since $|a_1b_1| = \Delta(R)$ and $|a_2b_2| = \Delta(R)$, we see that xz is a thickness chord of R. Thus from the definition of the width of a strip (see the introduction), we get that the disk G with center at the midpoint of xz and radius $\frac{1}{2} \cdot \Delta(R)$ is inscribed in Q and that $x, z \in \mathrm{bd}(G)$. Of course, also R is inscribed in Q and $x, z \in \mathrm{bd}(R)$. Hence, for every direction m between m_1 and m_2 the thickness chord xz of R has its endpoints on the lines bounding $S(R,m)$ and $S(G,m)$. So $w(R,m) = \Delta(R)$ for every direction m between m_1 and m_2. This contradicts the assumption of the theorem that the inequality $w(R,m) > \Delta(R)$ holds for every m strictly between m_1 and m_2. Consequently, a_1a_2 and b_1b_2 are not parallel.

Part 6: Since any degenerate segment is parallel to every segment, by Part 5 we see that a_1a_2 and b_1b_2 are non-degenerate. Since the segments a_1a_2 and b_1b_2 are not parallel and since they are in the sides of Q, one of them is perpendicular to m_1 and the other to m_2 (see Figures 3 and 4).

3. Corollaries

We intend to present a number of corollaries of the theorem which permit to describe the shape of reduced bodies in M^2. The proofs of the corollaries are given at the end of this section.

Corollary 1: *Let R be a reduced body of M^2 and let m_0 be a direction such that $w(R,m_0) = \Delta(R)$. Assume that the direction m_0 is an end-point of an open interval of directions m for which $w(R,m) > \Delta(R)$. Then exactly one of the lines bounding the strip $S(R,m_0)$ strictly supports R.*

This corollary generalizes statement (6) from [9]. The following corollary generalizes Theorem 2 of [9].

Corollary 2: *Let $w(R,m) = \Delta(R)$, for a reduced body $R \subset M^2$ and for a direction m. If the endpoints of a thickness chord of the unit ball D of M^2 through the origin are in the lines bounding $S(D,m)$ and if they are extreme points of D, then there is exactly one parallel thickness chord of R.*

The next corollary refers to parallel opposite segments in the boundary of a reduced body $R \subset M^2$.

Corollary 3: *Let $R \subset M^2$ be a reduced body and let $w(R,m) = \Delta(R)$ for a direction m. If each of the bases $A(R,m)$ and $B(R,m)$ of $T(R,m)$ is* *a degenerate segment, then the intersection of R with the union of the lines bounding $S(R,m)$ is $A(R,m) \cup B(R,m)$. If the straight line containing one of the bases, say $B(R,m)$, has a segment in common with R which properly contains $B(R,m)$, then $A(R,m)$ is a degenerate segment.*

Example: Corollary 3 is illustrated in Figure 5. In the left picture we see the unit ball D of M^2. It is the convex hull of two halves of a circle (obtained by a vertical cut) which are translated horizontally. We also see two segments through the origin o which connect opposite end-points of these half-circles. Translations of these segments are depicted in two reduced bodies $R_1, R_2 \subset M^2$ presented in the middle and right picture. Two horizontal supporting lines of R_1 have only the bases of the corresponding trapezium $T(R_1, m)$ in common with R_1. These bases are non-degenerate segments. The top horizontal supporting line of R_2 has a one-point base in common with R_2. The bottom horizontal supporting line has a segment that properly contains the opposite base of the corresponding trapezium $T(R_2, m)$ in common with R_2. By the way, R_1 does not contain butterflies, and R_2 contains exactly two butterflies which are marked in Figure 5 (for a definition of butterflies see below, after Corollary 10).

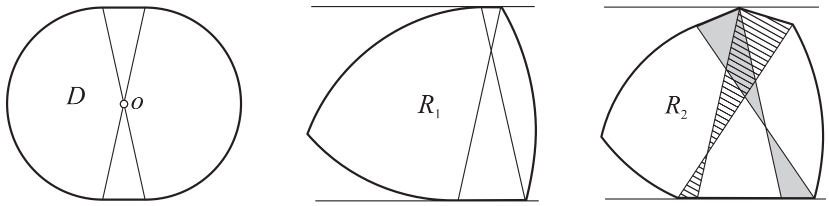

Figure 5

Corollary 4: *The sum of the lengths of the bases $A(R,m)$ and $B(R,m)$ is $\Delta(R)$ times the length of any of the boundary segments of the unit ball of M^2 parallel to them.*

Corollary 5: *If a reduced body $R \subset M^2$ has two parallel thickness chords which are in two straight lines of distance $\lambda \cdot \Delta(R)$, then the unit ball of M^2 has two thickness chords parallel to them, which are in straight lines of distance at least 2λ.*

Corollary 6: *If the boundary of a reduced body $R \subset M^2$ contains a segment, then the parallel R-strip has M^2-width $\Delta(R)$.*

Corollary 7: *Let $R \subset M^2$ be a reduced body. If at least one of the lines bounding an R-strip is an extreme supporting line of R, then the M^2-width of this strip is $\Delta(R)$.*

The next corollary is a strengthening of Theorem 1 in [9].

Corollary 8: *Let $R \subset M^2$ be a reduced body. Through every boundary point of R an R-strip of M^2-width $\Delta(R)$ passes.*

Corollary 9: *For every reduced body $R \subset M^2$ we have* $\mathrm{diam}(R)/\Delta(R) \leq 2$.

The Euclidean case of the following statement was first obtained in [3]. See also Corollary 1 in [9].

Corollary 10: *Every strictly convex reduced body of M^2 is a body of constant M^2-width.*

Consider the segments a_1a_2 and b_1b_2 as in the formulation of our theorem. Denote by c the intersection point of the segments a_1b_1 and a_2b_2. The union of the convex hull of $\{a_1, a_2, c\}$ and of the convex hull of $\{b_1, b_2, c\}$ is called the **butterfly determined by diagonals** a_1b_1 **and** a_2b_2 or, shortly, a **butterfly**. We call a_1a_2 and b_1b_2 **the arms** of the butterfly. Let us add that the idea of butterflies appears earlier in some considerations of [9], [11] and [13] without giving a definition. However, the last two papers use the colloquial term "butterfly".

Corollary 11: *The boundary of every reduced body $R \subset M^2$ is the union of the arms of all butterflies of R and of the endpoints of all thickness chords of R.*

Some of the above endpoints can be in many thickness chords. Of course, the diagonals of any butterfly are also thickness chords of R. We leave it to the reader to show that *the arms of two different butterflies of a reduced body $R \subset M^2$ can have at most endpoints of a thickness chord of R in common.*

Proof of the corollaries: We start with the proof of Corollary 1. The assumptions of the theorem are satisfied, where m_0 is m_1 or m_2. Let, for instance, $m_0 = m_1$. By the proof of Part 5 of the theorem, we have $a_1 = x$ and $b_2 = z$ like in Figure 4 (or we have $a_2 = x$ and $b_1 = z$ like in Figure 3; then the further considerations are analogous). We see that b_1b_2 is a non-degenerate segment perpendicular to m_1. Hence one of the two lines bounding $S(R, m_0)$ contains more than one point of R. It remains to show that the other line bounding $S(R, m_0)$ strictly supports R, i.e., that it contains exactly one point of R. In other words, it is sufficient to show that $\mathrm{bd}(R)$ and the segment wx have only the point $a_1 = x$ in common. Suppose the opposite. Then there is a point $p \in wx$ such that $p \in \mathrm{bd}(R)$ and $p \neq a_1$. Consider the straight lines parallel to the segment a_1b_1 which pass through p and through b_2. Observe that at least one of these lines has non-empty intersection with both segments b_1b_2 and pa_1. Denote this line by W. Let q be the point of intersection of W and pa_1, and r be the point of intersection of W and b_1b_2. From the description of W we obtain $r \neq b_1$. Since $|a_1b_1| = \Delta(R)$, and since qr is a translate of a_1b_1, we obtain $|qr| = \Delta(R)$. This contradicts the assumption of the theorem that b_1 is the last point of $B(R, m_1)$.

The proof of Corollary 2 is analogous to Part III of the proof of Theorems 2 and 3 from [9]. Again we assume the opposite, i.e., we assume that in R there are two different thickness chords parallel to the thickness chord (mentioned in Corollary 2) of the unit ball D. The convex hull of the union of them is a parallelogram Z. Observe that the M^2-distance of the endpoints of a diagonal of Z is larger than $\Delta(R)$. This permits to apply Corollary 1, yielding a contradiction to $Z \subset R$.

Corollary 3 follows immediately from Corollary 2.

Let us show Corollary 4. We apply Corollary 2. Observe that the lines bounding $S(D, m)$ and the unit ball D have two segments in common which are symmetric with respect to the origin (these segments can also be degenerate). Thus the endpoints of these segments are just the extreme points of D considered in the formulation of Corollary 2. Hence we conclude the thesis of Corollary 4.

Corollary 5 is an easy result of Corollary 4.

The proof of Corollary 6, applying our theorem, can be done similarly to the proof of part (a) of Theorem 4 from [9]. We leave this task to the reader.

The proof of Corollary 7, based on our Corollary 6, is analogous to the first part of the proof of Theorem 5 in [9].

Corollary 8 follows immediately from Corollary 7.

For the proof of Corollary 9 it suffices to show that $|xy| \leq 2 \cdot \Delta(R)$ for every $x, y \in \mathrm{bd}(R)$. By Corollary 8 there are $x', y' \in \mathrm{bd}(R)$ such that $|xx'| = |yy'| = \Delta(R)$. The segments xx' and yy' connect opposite bounding lines of the corresponding strips from Corollary 8. If they intersect, the triangle inequality implies $|xy| \leq 2 \cdot \Delta(R)$. If these segments do not intersect, then there is a direction m for which $x, y \in A(R, m)$ and $x', y' \in B(R, m)$. From Corollary 4 we conclude that $|xy| \leq 2 \cdot \Delta(R)$.

In order to see Corollary 10, we proceed in a similar way as in the proof of Corollary 1 in [9]. Just assume that there is a reduced body $R \subset M^2$ which is not of constant M^2-width. By the theorem, $\mathrm{bd}(R)$ contains a segment. Thus R is not strictly convex, a contradiction.

Corollary 11 is an immediate consequence of our theorem. ∎

The converse implication to the one given in Corollary 7 is not true; just take a normed plane whose unit ball is a square, and identify this square with R. Corollaries 7 and 8 are not true for $d \geq 3$, even in Euclidean space, as the example presented in Figure 2 of [10] shows. The same example also shows that no universal finite estimate as in Corollary 9 can be formulated for dimensions $d \geq 3$. Moreover, the value 2 in Corollary 9 cannot be improved. We get an example by taking a square as unit ball and R as the convex hull of the center and of two neighbouring vertices of this square.

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